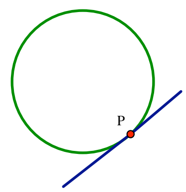
Lecture 2 : Tangents

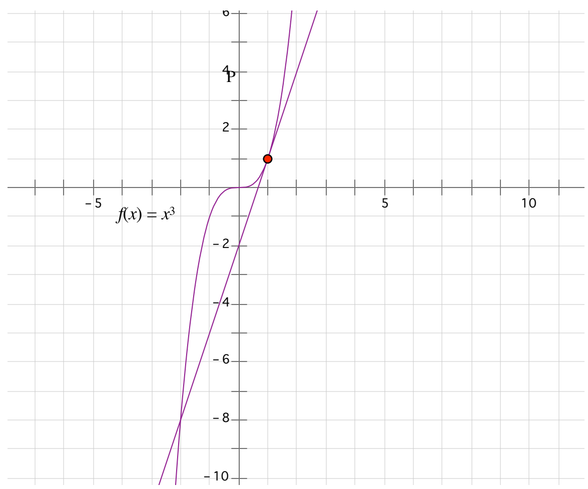
Functions

The word Tangent means "touching" in Latin. The idea of a tangent to a curve at a point P, is a natural one, it is a line that touches the curve at the point P, with the same direction as the curve. However this description is somewhat vague, since we have not indicated what we mean by the direction of the curve.

In Euclidean Geometry, the notion of a tangent to a circle at a point P on its circumference, is precise; it is defined as the unique line through the point P that intersects the circle once and only once.



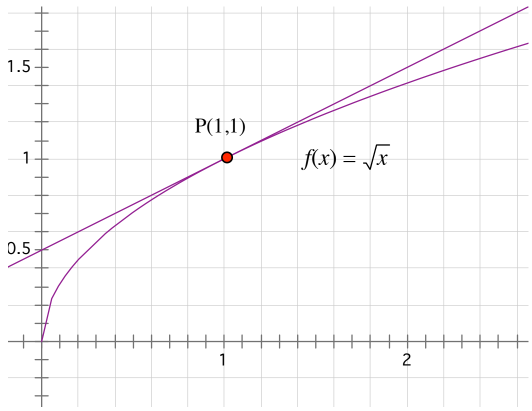
This definition works for a circle, however in the case of the curve shown below, the object we wish to use as the tangent line intersects the curve more than once. In other cases there may not be a unique line touching the curve at a point. We must therefore make this intuitive definition of the tangent more precise.



Over the course of the next two chapters (Ch 2 and 3), we will make precise, what we mean by the slope or direction of a curve at a point P**.** We will accomplish a definition, and a method of measuring the slope, with the aid of the concept of a limit. We will then use our measure of the slope of the curve at a point P (when it exists) to define the tangent at the point P as the line through P with the same slope as the curve at that point.

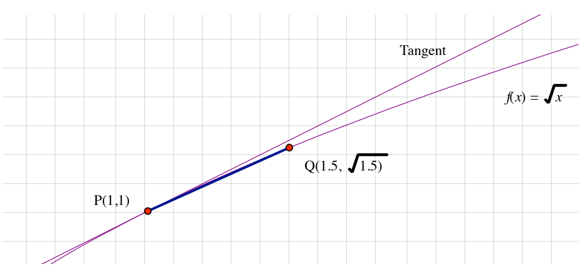
Although the process of defining the slope and learning to calculate slopes (derivatives) for a wide range of functions will take some time, we can see the concept in action immediately with some examples.

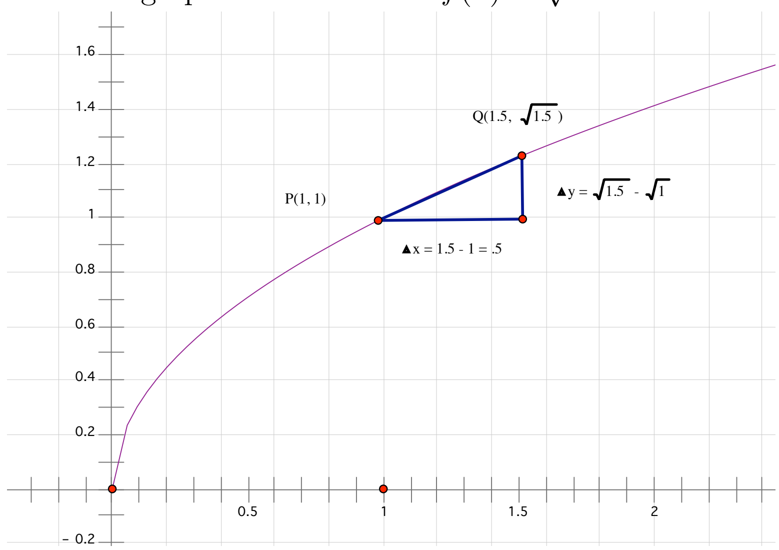
Example 1 Find the equation of the tangent line to the curve $y = \sqrt{x}$ at the point where $x = 1$ (at the point $P(1,1)$). This means, we need to find the slope of the tangent line touching the curve drawn in the picture.



We have only one point on the tangent line, $P(1,1)$, which is not enough information to find the slope. However **we can approximate the slope of this line using the slope of a line segment joining $P(1,1)$ to a point Q on the curve near P.**

Let us consider the point $Q(1.5, \sqrt{1.5})$, which is on the graph of the function $f(x) = \sqrt{x}$.





Since Q is on the curve $y = \sqrt{x}$, the slope of the line segment joining the points P and Q (secant),

$$m_{PQ} = \frac{\Delta y}{\Delta x} = \frac{\sqrt{1.5} - \sqrt{1}}{1.5 - 1} \approx$$

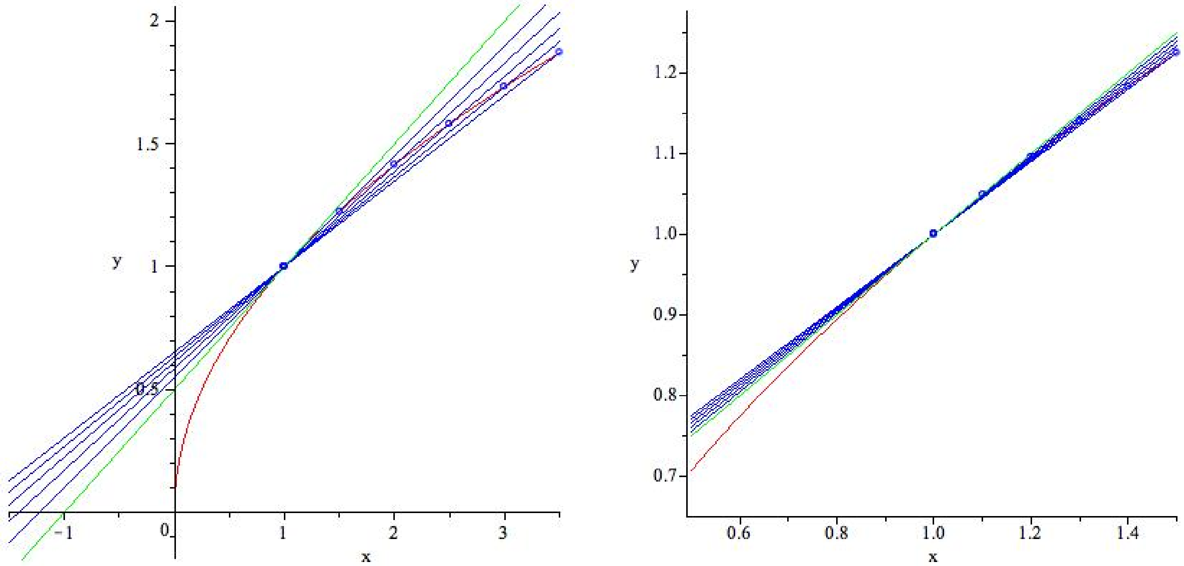
m_{PQ} the change in elevation on the curve $y = \sqrt{x}$ between the points P and Q divided by the change in the value of x, $\frac{\Delta y}{\Delta x}$ (see diagram on right). If we think of the curve $y = \sqrt{x}$ as a hill and imagine we are walking up the hill from left to right, m_{PQ} agrees with our intuitive idea of the average slope or incline on the hill between the points P and Q.

Because, the point Q is so close to P, and because the curve $y = \sqrt{x}$ does not deviate too far from its tangent near P,

slope of tangent at the point $P \approx m_{PQ} =$

If we choose a point Q on the curve $y = \sqrt{x}$ which is closer to the point P and calculate

the slope of the line segment PQ, m_{PQ}, we get a better estimate (in this case) **for the slope of the tangent line to the curve at P.** Fill in the table below to see what happens to the slopes of the secants PQ as the point Q moves closer to P



	slope of secant$(Q = Q(x, \sqrt{x}))$	Δx	Δy
x	$m_{PQ} = \frac{\sqrt{x}-\sqrt{1}}{x-1} = \frac{\text{Change in y (from P to Q)}}{\text{Change in x (from P to Q)}}$	$x - 1$	$\sqrt{x} - \sqrt{1}$
3.5	$\frac{\sqrt{3.5}-1}{2.5} = .348$	2.5	.8708
3.0	$\frac{\sqrt{3}-1}{2} = .366$	2	.7320
2.5	$\frac{\sqrt{2.5}-1}{1.5} = .387$	1.5	.5811
2.0	$\frac{\sqrt{2}-1}{1} = .414$	1	.414
1.5	$\frac{\sqrt{1.5}-1}{.5} = .449$	.5	.2247
1.2	.4772	.2	.0954
1.1	.4881	.1	.0488
1.01	.4987	.01	.00498
1.001	$\frac{\sqrt{1.001}-1}{.001} =$	.001	4.99×10^{-4}
1.0001		.0001	
1.00001		.00001	

Fill in the last few lines of the table and complete the following sentence:
As x approaches 1, the values of m_{PQ} approach ________

We can rephrase the sentence on the previous page using Δx and Δy :
As Δx approaches 0, the values of m_{PQ} approach 1/2
or :

$$\boxed{\text{As } \Delta x \text{ approaches 0, the values of } \frac{\Delta y}{\Delta x} \text{ approach } 1/2}$$

We can rewrite this statement (to which we will assign a precise meaning later) in a number of ways, all of which will be used in the course. We say:

$$\boxed{\lim_{x \to 1} m_{PQ} = 1/2}$$

or

$$\lim_{Q \to P} m_{PQ} = 1/2$$

or

$$\boxed{\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = 1/2.}$$

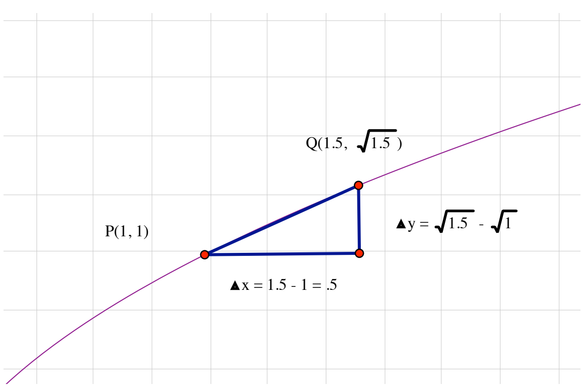
From the picture above, we can see that the slopes of the line segments PQ approach the slope of the tangent we seek, as Q approaches P. Hence it is reasonable to define the slope of the tangent to be this limit of the slopes of the line segments PQ as Q approaches P. This will be called the derivative of the function $f(x) = \sqrt{x}$ at $x = 1$ later and will be denoted by $f'(1)$.

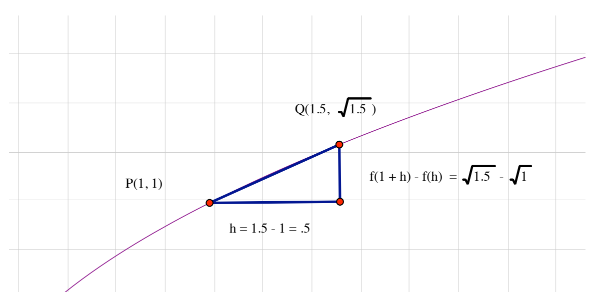
Hence the slope of the tangent to the curve $y = \sqrt{x}$ at the point $P(1, 1)$ is 1/2 and the equation of the tangent to the curve $y = \sqrt{x}$ at this point is

$$\text{Equation of the tangent at } \quad P \text{ is } \quad \boxed{y - 1 = \frac{1}{2}(x - 1)} \quad \text{ or } \quad y = \frac{1}{2}x + \frac{1}{2}.$$

We will also make heavy use of the following notation: When calculating the slope of a secant, instead of using Δx to denote the small change in the value of x (between P and Q), we use h. For P and Q shown in the diagrams below, this translates to

$$m_{PQ} = \frac{\sqrt{1+h} - \sqrt{1}}{h} = \frac{\sqrt{1.5} - \sqrt{1}}{.5}$$





When we rewrite our table replacing Δx by h, we see that we can rephrase our statement about the limiting value of the slope of the secants as

$$\boxed{\text{As } h \text{ approaches 0, the values of } m_{PQ} = \frac{\sqrt{1+h}-\sqrt{1}}{h} \text{ approach } 1/2}$$

or in the language of limits :

$$\boxed{\lim_{h\to 0} m_{PQ} = \lim_{h\to 0} \frac{\sqrt{1+h}-\sqrt{1}}{h} = 1/2}$$

	slope of secant$(Q = Q(x, \sqrt{x}))$	h	$f(1+h) - f(1)$
x	$m_{PQ} = \frac{\sqrt{1+h}-\sqrt{1}}{h} = \frac{\text{Change in y (from P to Q)}}{\text{Change in x (from P to Q)}}$	h	$\sqrt{1+h} - \sqrt{1}$
3.5	$\frac{\sqrt{3.5}-1}{2.5} = .348$	2.5	.8708
3.0	$\frac{\sqrt{3}-1}{2} = .366$	2	.7320
	$\vdots$	$\vdots$	
1.00001		.00001	.499999987

The slope of the tangent to a curve at a point gives us a measure of the **instantaneous rate of change** of the curve at that point. This measure is not new to us, in a car, the odometer tells us the distance the car has travelled (under its own steam) since it rolled off the assembly line. This a function D of time, t. The speedometer on a car gives us the instantaneous rate of change of the function $D(t)$, with respect to time, t, at any given time. When you are driving a car, you see that the **speed** of the car is usually changing from moment to moment. This reflects the fact that the instantaneous rate of change of $D(t)$ or slope of the tangent to the curve $y = D(t)$ varies from moment to moment.

The following Wolfram demonstration shows the above process in action for a wider variety of examples:

Increasing/Decreasing Functions

When a function is increasing, we get a **positive slope** for the tangent and when a function is decreasing, we get a **negative slope** for the tangent. $D(t)$ above never decreases, reflecting the fact that the speedometer always reads 0 or something positive.

Example A Buzz Lightyear toy is dropped (no initial velocity) from the top of the Willis Tower in Chicago, which is 442 m tall.

We will denote **the distance fallen by the toy after t seconds by $s(t)$ meters** . According to Gallileo's law, the distance fallen by any free falling object is proportional to the square of the time it has been falling. Hence, $s(t) = kt^2$. Let us assume that the only force acting on the toy is the force of gravity (no air resistance or wind) which causes the speed of the toy to increase as it falls with an acceleration of $9.8m/s^2$ or $32ft/s^2$. Later we will see that this translates to saying that distance fallen by the toy after t seconds is

$$s(t) = 4.9t^2 \text{ meters}$$

The velocity or speed of the toy at any given time is the instantaneous rate of change of the function $s(t)$ at that time.

(a) How far has the toy travelled after $t = 3$ seconds?

(b) How long does it take for the toy to reach the ground?

(c) What is the average speed of the toy on its way to the ground?

(d) Use the table below to estimate the velocity of the toy after 3 seconds?

Time Interval	Average velocity $= \frac{\Delta s}{\Delta t}$ (measured in m/s)
$3 \leq t \leq 4$	
$3 \leq t \leq 3.1$	
$3 \leq t \leq 3.01$	
$3 \leq t \leq 3.001$	
$3 \leq t \leq 3.0001$	

EXTRAS: Attempt the following questions before you look at the solutions provided

In the following examples, we have empirical data about a function, rather than a formula for the function. However the methods of estimation outlined above rely only on knowing the values of the function at a finite number of points. Therefore the same method can be used to estimate instantaneous rate of change from a finite set of data:

Example The following data shows the position of a sprinter, $s(t)$ = meters travelled after t seconds.

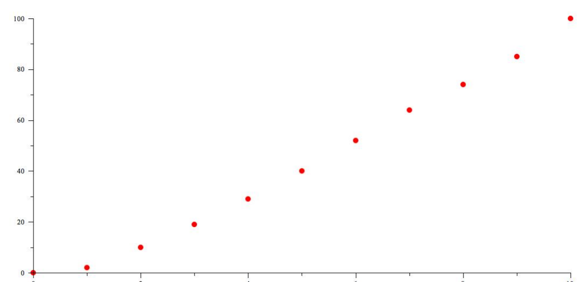
t (*seconds*)	0	1	2	3	4	5	6	7	8	9	10
$s(t)$ (*meters*)	0	2	10	19	29	40	52	64	74	85	100

(a) Find the average velocity of the sprinter over the time periods $[1, 8]$, $[1, 3]$ and $[1, 2]$.

(b) Which of the above averages gives the best estimate of the instantaneous velocity of the sprinter when $t = 1$?

(c) Can you think of any other way to estimate the instantaneous velocity of the sprinter when $t = 1$?

It may help to consider the graph of the data



Example The following data shows the world population in the 20th century.

Year	Population (in millions)	Year	Population (in millions)
1900	1650	1960	3040
1910	1750	1970	3710
1920	1860	1980	4450
1930	2070	1990	5280
1940	2300	2000	6080
1950	2560		

Estimate the rate of population growth in 1920 and compare it with an estimate of the rate of population growth in 1980. (Note you have a number of choices for your estimates).

Solutions

Example The following data shows the position of a sprinter, $s(t)$ = meters travelled after t seconds.

t (*seconds*)	0	1	2	3	4	5	6	7	8	9	10
$s(t)$ (*meters*)	0	2	10	19	29	40	52	64	74	85	100

(a) Find the average velocity of the sprinter over the time periods $[1, 8]$, $[1, 3]$ and $[1, 2]$.

The average velocity from $t = 1$ to $t = 8$ is given by

$$\frac{s(8) - s(1)}{8 - 1} = \frac{74 - 2}{7} = 10.28 \text{ m/s}.$$

The average velocity from $t = 1$ to $t = 3$ is given by

$$\frac{s(3) - s(1)}{3 - 1} = \frac{19 - 2}{2} = 8.5 \text{ m/s}.$$

The average velocity from $t = 1$ to $t = 2$ is given by

$$\frac{s(2) - s(1)}{2 - 1} = \frac{10 - 2}{1} = 8 \text{ m/s}.$$

(b) Which of the above averages gives the best estimate of the instantaneous velocity of the sprinter when $t = 1$?

The average velocity for the sprinter over the time period $[1, 2]$ is most likely to give us the best estimate for the instantaneous velocity of the sprinter when $t = 1$, since the change in velocity over that period is likely to be less because it is the smallest time interval.

(c) Can you think of any other way to estimate the instantaneous velocity of the sprinter when $t = 1$?

Another good estimate for the instantaneous velocity at $t = 1$ is the average velocity between $t = 0$ and $t = 1$;

$$\frac{s(1) - s(0)}{1 - 0} = \frac{2 - 0}{1} = 2 \text{ m/s}.$$

Also another good estimate for the instantaneous velocity at $t = 1$ is the average velocity between $t = 0$ and $t = 2$;

$$\frac{s(2) - s(0)}{2 - 0} = \frac{10 - 0}{2} = 5 \text{ m/s}.$$

Example The following data shows the world population in the 20th century.

Year	Population (in millions)	Year	Population (in millions)
1900	1650	1960	3040
1910	1750	1970	3710
1920	1860	1980	4450
1930	2070	1990	5280
1940	2300	2000	6080
1950	2560		

Estimate the rate of population growth in 1920 and compare it with an estimate of the rate of population growth in 1980. (Note you have a number of choices for your estimates).

To estimate the rate of growth in 1920, we take the average rate of growth per year between 1910 and 1930:

$$\text{rate of growth in 1920} \approx \frac{2070 - 1750}{1930 - 1910} = 16 \text{ million/year}.$$

To estimate the rate of growth in 1980, we take the average rate of growth per year between 1970 and 1990:

$$\text{rate of growth in 1980} \approx \frac{5280 - 3710}{1990 - 1970} = 78.5 \text{ million/year.}$$